Introduction to Mathematical Quantum Theory

Text of all the Exercises

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Contents

1 Exercise Sheet 1

1.1 Exercise 1 - Examples of Fourier transforms

a Consider the function $f \in L^1(\mathbb{T})$ defined as the periodization of

$$f(x) := x(2\pi - x). \tag{1}$$

Calculate the Fourier coefficients of f and use them to prove that

$$\sum_{k=0}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{2}$$

b Let σ be a positive real number and $\mathbf{v},\ \mathbf{u} \in \mathbb{R}^d$. Consider the function $g_{\sigma,\mathbf{v},\mathbf{u}}$ in the space $L^2\left(\mathbb{R}^d\right)$ with $d \in \mathbb{N}$ defined as

$$g_{\sigma,\mathbf{v},\mathbf{u}}(\mathbf{x}) := \left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}. \tag{3}$$

Then prove that $\widehat{g}_{\sigma,\mathbf{v},\mathbf{u}} = e^{i\mathbf{v}\cdot\mathbf{u}} g_{\sigma^{-1},\mathbf{u},-\mathbf{v}}$, i.e.

$$\mathcal{F}\left[\left(\frac{\sigma}{\pi}\right)^{\frac{d}{4}} e^{-\frac{\sigma}{2}|\mathbf{x}-\mathbf{v}|^2 + i\mathbf{u}\cdot\mathbf{x}}\right](\mathbf{k}) = \left(\frac{1}{\sigma\pi}\right)^{\frac{d}{4}} e^{-\frac{1}{2\sigma}|\mathbf{k}-\mathbf{u}|^2 - i\mathbf{u}\cdot(\mathbf{k}-\mathbf{v})}. \tag{4}$$

1.2 Exercise 2 - Properties of operator norm and definition of boundedness (complement to the exercise session)

Consider V_1 and V_2 two normed vector spaces over[1] $\mathbb{F}$ and $T : V_1 \to V_2$ a linear mapping. Define $\|T\|_{V_1,V_2}$ as

$$\|T\| := \sup_{v \in V_1,\ v \neq 0} \frac{\|Tv\|}{\|v\|}. \tag{5}$$

[1] Here and in the following $\mathbb{F}$ can be chosen to be either $\mathbb{R}$ or $\mathbb{C}$.

For a generic linear mapping T we have $\|T\| \in [0, +\infty]$. Prove that

$$\|T\| = \sup_{v \in V_1,\ \|v\|_{V_1} = 1} \|Tv\| \tag{6}$$

$$= \sup_{v \in V_1,\ \|v\|_{V_1} \leqslant 1} \|Tv\|. \tag{7}$$

Prove moreover that the following are equivalent

a T is continuous.

b T is continuous in 0, meaning that for any sequence $\{v_n\}_{n \in \mathbb{N}} \subseteq V_1$,

$$v_n \to 0 \quad \Longrightarrow \quad Tx_n \to 0. \tag{8}$$

c The quantity $\|T\|$ is finite, meaning that $\|T\| < +\infty$.

1.3 Exercise 3 - Young Inequality

Consider $p,\ q,\ r \in [1, +\infty]$ such that

$$\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{p}. \tag{9}$$

Let $f \in L^q\left(\mathbb{R}^d\right)$, $g \in L^r\left(\mathbb{R}^d\right)$; prove that

$$\|f * g\|_p \leqslant \|f\|_q \|g\|_r. \tag{10}$$

Hint: Consider the functions $\alpha,\ \beta,\ \gamma$ defined as

$$\alpha(\mathbf{x}, \mathbf{y}) := |f(\mathbf{y})|^q |g(\mathbf{x} - \mathbf{y})|^r, \tag{11}$$

$$\beta(\mathbf{y}) := |f(\mathbf{y})|^q, \tag{12}$$

$$\gamma(\mathbf{x}, \mathbf{y}) := |g(\mathbf{x} - \mathbf{y})|^r, \tag{13}$$

notice that

$$|f * g(\mathbf{x})| \leqslant \int_{\mathbb{R}^d} \alpha(\mathbf{x}, \mathbf{y})^{\frac{1}{p}} \beta(\mathbf{y})^{\frac{p-q}{pq}} \gamma(\mathbf{x}, \mathbf{y})^{\frac{p-r}{pr}} \, d\mathbf{y} \tag{14}$$

and that

$$\frac{1}{p} + \frac{p-q}{pq} + \frac{p-r}{pr} = 1 \tag{15}$$

to apply Hölder inequality.

1.4 Exercise 4 - Fourier transform and sinc

a Prove that there exists a positive real number C such that we have

$$\sup_{0\leqslant a<b<+\infty}\left|\int_a^b \frac{\sin x}{x}dx\right| \leqslant C. \tag{16}$$

Hint: Consider the function

$$F(t) := \int_0^\eta e^{-tx}\frac{\sin x}{x}dx. \tag{17}$$

Deduce a bound on $F'(t)$ uniform in η. Apply the fundamental theorem of calculus for $F(0)$ to conclude.

b Consider an odd function $f \in L^1(\mathbb{R})$. Prove that for any such function we have

$$\sup_{0\leqslant a<b<+\infty}\left|\int_a^b \frac{\widehat{f}(k)}{k}dk\right| \leqslant \frac{C}{(2\pi)^{\frac{d}{2}}}\|f\|_1. \tag{18}$$

c Let $g(k)$ be a continuous odd function on the line such that is equal to $1/\log k$ for any $k \geqslant 2$. Prove that there cannot be an $L^1(\mathbb{R})$ function whose Fourier transform is g.

2 Exercise Sheet 2

2.1 Exercise 1 - Fourier transform and convolution

Let $f,\ g \in \mathscr{S}\left(\mathbb{R}^d\right)$. Recall that in class we proved

$$\widehat{f * g} = (2\pi)^{\frac{d}{2}}\, \widehat{f}\widehat{g}. \tag{19}$$

Prove that

$$\widehat{f} * \widehat{g} = (2\pi)^{\frac{d}{2}}\, \widehat{fg}. \tag{20}$$

Hint: Consider the equivalent statement of (19) *for the inverse of the Fourier transform and apply it to* $\widehat{fg}$.

2.2 Exercise 2 - Unique projector (complement to the class)

Let $\mathcal{H}$ be an Hilbert space and V a closed linear subspace of $\mathcal{H}$.

a In class we proved that for any $f \in \mathcal{H}$ there exists an element $g_f \in V$ such that

$$\|f - g_f\| = \min_{h \in V} \|f - h\|\,. \tag{21}$$

Prove that g_f is the unique element of V that satisfies the minimum.

b In class we proved that g_f is such that $f - g_f \in V^{\perp}$. Prove that there is no other element $h \in V$ such that $f - h \in V^{\perp}$.

2.3 Exercise 3 - Hilbert space basis with Hahn-Banach

Let $\mathcal{H}$ be an Hilbert space. Prove that there exists a basis for $\mathcal{H}$. Prove moreover that $\mathcal{H}$ is separable if and only if there exists a countable base for it.

Hint: For the first part apply Zorn's Lemma to the set of (also infinite) orthonormal systems ordered by inclusion. Prove that any maximal orthonormal system is a base, i.e. is dense.

For the second part prove and use the following fact: if f *is an element of* $\mathcal{H}$ *and* S *is a basis for* $\mathcal{H}$*, there exists a sequence of elements* $\{e_n\}_{n\in\mathbb{N}} \subseteq S$ *such that* $f \in \overline{\operatorname{span}_{\mathbb{K}} \{e_n\}_{n\in\mathbb{N}}}$.

2.4 Exercise 4 - Property of the adjoint (bounded operators)

Let $A,\ B$ bounded operators on an Hilbert space $\mathcal{H}$ and $\alpha,\ \beta \in \mathbb{C}$. Prove the following equalities:

$$\mathrm{id}^* = \mathrm{id} \tag{22}$$

$$(A^*)^* = A \tag{23}$$

$$(AB)^* = B^* A^* \tag{24}$$

$$(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*. \tag{25}$$

Moreover, prove that A^* is bounded and that $\|A^*\| = \|A\|$.

3 Exercise Sheet 3

3.1 Exercise 1 - Properties of orthogonal projectors

Let $\mathcal{H}$ be a Hilbert space. Let V any closed subspace of $\mathcal{H}$; recall the definition of $V^{\perp}$ as

$$V^{\perp} := \{f \in \mathcal{H} \mid \langle g, f\rangle = 0 \ \forall g \in V\}. \tag{26}$$

We saw in class that the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = V \oplus V^{\perp}$, meaning that $V \cap V^{\perp} = \{0\}$ and that for any non-zero $f \in \mathcal{H}$ there exists a unique element $f_V \in V$ such that $f - f_V \in V^{\perp}$. Define $P_V f := f_V$; from the uniqueness of f_V this is a well defined linear mapping.

- **a** Prove that $P_V^2 = P_V = P_V^*$.
- **b** Use **a** to prove that P_V is bounded and if $V \neq \{0\}$ then $\|P_V\| = 1$.
- **c** Prove that if V_1 and V_2 are two closed subspaces of $\mathcal{H}$ then[2]

$$V_1 \perp V_2 \qquad \Longleftrightarrow \qquad P_{V_1} P_{V_2} = 0. \tag{27}$$

3.2 Exercise 2 - Derivative of inner product (complement to the class)

Let $\phi(t)$ and $\psi(t)$ differentiable functions on the Hilbert space $\mathcal{H}$, meaning that the limit

$$\frac{d\phi}{dt}(t) := \lim_{h\to 0} \frac{\phi(t+h) - \phi(t)}{h} \tag{28}$$

exists in the norm topology of $\mathcal{H}$ for each $t \in \mathbb{R}$, and similarly for $\psi(t)$.

Prove that

$$\frac{d}{dt}\langle \phi(t), \psi(t)\rangle = \langle \frac{d\phi}{dt}(t), \psi(t)\rangle + \langle \phi(t), \frac{d\psi}{dt}(t)\rangle \tag{29}$$

3.3 Exercise 3 - $\frac{1}{i\hbar}[A, B]$ is self-adjoint

Let $\mathcal{H}$ be a Hilbert space. Consider A and B bounded self-adjoint operators on $\mathcal{H}$. Prove that $\frac{1}{i\hbar}[A, B]$ is self adjoint.

3.4 Exercise 4 - Properties of the commutator

Consider a vector space V over $\mathbb{C}$, $A,\ B,\ C$ linear bounded operators on V and $\alpha \in \mathbb{C}$.

- **a** Prove that $[A, B + \alpha C] = [A, B] + \alpha [A, C]$.

[2] We denote with $\perp$ the condition of two subspaces of an Hilbert space $\mathcal{H}$ of being orthogonal, i.e., V_1 is orthogonal to V_2, or $V_1 \perp V_2$ if and only if for any $(f, g) \in V_1 \times V_2$ we have $\langle f, g\rangle = 0$.

b Prove that $[B, A] = -[A, B]$.

c Prove that $[A, BC] = [A, B]\,C + B\,[A, C]$.

d Prove that $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$.

4 Exercise Sheet 4

4.1 Exercise 1 - Two bounded operator cannot commute in a nontrivial manner

Let $\mathcal{H}$ be an Hilbert space. Let A and B linear operators on $\mathcal{H}$ such that there exists $\alpha \in \mathbb{C}\backslash\{0\}$ such that

$$[A, B] = \alpha \,\mathrm{id}\,. \tag{30}$$

Prove that A and B cannot be both bounded.

Hint: Assume both bounded; consider $\|[A, B^n]\|$ and find an absurd.

4.2 Exercise 2 - Fourier transform of the complex gaussian

a Prove that for any $\alpha \in \mathbb{C}$ such that $\mathrm{Re}\,(\alpha) > 0$,

$$\left(\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx\right)^2 = \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2\alpha}} dxdy \tag{31}$$

$$= 2\pi\alpha, \tag{32}$$

where the integral over $\mathbb{R}^2$ can be evaluated using polar coordinates. Deduce that

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2\alpha}} dx = \sqrt{2\pi\alpha}, \tag{33}$$

where the square root is the one with positive real part.

b For all $B \geqslant A > 0$ and $\alpha \in \mathbb{C}\backslash\{0\}$ we have

$$\int_A^B e^{-\frac{x^2}{2\alpha}} dx = -\frac{\alpha}{x} e^{-\frac{x^2}{2\alpha}}\bigg|_A^B - \int_A^B \frac{\alpha}{x^2} e^{-\frac{x^2}{2\alpha}} dx. \tag{34}$$

Using this, prove that the integral in (33) is convergent for all nonzero α with $\mathrm{Re}\,(\alpha) \geqslant 0$, provided the integral is interpreted as a principle value when not absolutely convergent, where the principal value is defined as

$$\mathrm{PV}\int_{\mathbb{R}} f(x)\, dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{35}$$

c Prove that the result of **a** is also valid for nonzero values of α with $\mathrm{Re}\,(\alpha) = 0$, at least in the principal value.

Hint: Given $\eta \neq 0$, show that the principal value from A to $+\infty$ of $\exp\left[-\frac{x^2}{2(\gamma+i\eta)}\right]$ is small for large A, uniformly in $\gamma \in [0, 1]$.

d Prove that

$$\frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} e^{ikx} e^{-i\frac{\hbar t}{2m}k^2} dk = \sqrt{\frac{m}{2\pi i\hbar t}}\, e^{i\frac{m}{2\hbar t}x^2}, \tag{36}$$

where the square root is the one with real positive part.

4.3 Exercise 3 - Counterexample for the closed graph theorem

Consider a separable Hilbert space $\mathcal{H}$ and a complete orthonormal system for it $\{\varphi_n\}_{n\in\mathbb{N}}$. Assume that φ_∞ cannot be written as a finite linear combination of elements of $\{\varphi_n\}_{n\in\mathbb{N}}$. Let D denote the dense linear subspace of $\mathcal{H}$ consisting of all finite linear combinations of elements of $\{\varphi_n\}_{n\in\mathbb{N}}$ and of φ_∞. On D define the operator $T : D \to \mathcal{H}$ defined as

$$T\left(\alpha_\infty\varphi_\infty + \sum_{n\in\mathbb{N}} \alpha_n\varphi_n\right) := \alpha_\infty\varphi_\infty. \tag{37}$$

Prove that T is not bounded.

Hint: Use the closed graph theorem.

4.4 Exercise 4 - Free Schrödinger equation preserves the domain

Recall the definition of $H^2(\mathbb{R})$ as

$$H^2(\mathbb{R}) := \left\{\psi \in L^2(\mathbb{R}) \mid k^2\widehat{\psi} \in L^2(\mathbb{R})\right\}$$

Recall that in class we defined the map that to any initial datum $\psi_0 \in L^2(\mathbb{R})$ would associate $\psi_t := \widetilde{U}_0(t)\psi_0$, defined via the Hamiltonian $H_0 := -\frac{\partial^2}{\partial x^2}$ with domain $\mathcal{D}(H_0) = H^2(\mathbb{R})$. Indeed if $U_0(t)\psi_0$ is defined for any $\psi_0 \in \mathcal{S}(\mathbb{R})$ as the unique solution to

$$\begin{cases} i\hbar\partial_t\left(U_0(t)\psi_0\right) = H_0U_0(t)\psi_0 \\ U_0(t)\psi_0|_{t=0} = \psi_0, \end{cases} \tag{38}$$

then $\widetilde{U}_0(t)$ is defined by density on the whole space $L^2(\mathbb{R})$, and coincides with $U_0(t)$ on $\mathcal{S}(\mathbb{R})$.

Prove that if $\psi_0 \in D(H_0)$ then $\psi_t \in \mathcal{D}(H_0)$.

5 Exercise Sheet 5

5.1 Exercise 1 - Well-posedness of standard deviation

Let ψ be a unit vector in $L^2(\mathbb{R})$ such that $x\psi,\ x^2\psi \in L^2(\mathbb{R})$. Prove that

$$\langle X^2\rangle_\psi \geqslant (\langle X\rangle_\psi)^2\,, \tag{39}$$

where as we defined in class, X is the operator given by the multliplication by x and

$$\langle A\rangle_\psi := \langle \psi, A\psi\rangle. \tag{40}$$

Hint: Use Jensen inequality.

5.2 Exercise 2 - Operator norm of multiplication by a sequence

Let $\alpha := \{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence of complex numbers. Consider the Hilbert space of the square integrable functions $\mathfrak{h} := l^2(\mathbb{Z})$. Consider the operator that to the sequence $x := \{x_n\}_{n\in\mathbb{Z}}$ associate the sequence $M_\alpha x = \{\alpha_n x_n\}_{n\in\mathbb{Z}}$.

Suppose that $\|\alpha\|_\infty := \sup_{n\in\mathbb{Z}} |\alpha_n| < +\infty$. Prove that M_a is a well defined linear bounded operator from $\mathfrak{h}$ to itself and prove that $\|M_\alpha\| = \|\alpha\|_\infty$.

5.3 Exercise 3 - No solutions for too low energy in the potential well (complement to the class)

Consider the Hilbert space $\mathfrak{h} := L^2(\mathbb{R})$. And the operator H define

$$\begin{aligned}\mathcal{D}(H) &:= H^2(\mathbb{R}) = \left\{\psi \in L^2(\mathbb{R}) \mid k^2\widehat{\psi} \in L^2(\mathbb{R})\right\}\\ H &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(X)\,,\end{aligned}$$

where the operator $(V(X)\psi)(x) = V(x)\psi(x)$, with

$$V(x) := \begin{cases} -C & \text{if } |x| \leqslant A,\\ 0 & \text{if } |x| > A,\end{cases} \tag{41}$$

and with A and C positive constants. Consider $E \in (-\infty, -C]$ and prove that there is no nonzero $\psi_E \in \mathcal{D}(H)$ such that

$$H\psi_E = E\psi_E. \tag{42}$$

5.4 Exercise 4 - Odd solutions to the potential well (complement to the class)

Let $\mathfrak{h}$, H and $\mathcal{D}(H)$ as in Exercise 3. In class we saw that for any $E \in (-C, 0)$ there is always at least one nonzero even solution ψ_E to the problem $H\psi_E = E\psi_E$.

Prove that if $A\sqrt{2mC}\hbar \leqslant \frac{\pi}{2}$ there are no nonzero odd solutions, and for larger values of C there is always at least one.

6 Exercise Sheet 6

6.1 Exercise 1 - A preserves a space, A^* preserves the orthogonal

Let V be a closed subspace of $\mathcal{H}$ Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ such that $A(V) \subseteq V$. Prove that $A^*\left(V^\perp\right) \subseteq V^\perp$.

6.2 Exercise 2 - Inverse of the adjoint of an invertible

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear bounded operator on $\mathcal{H}$ with linear bounded inverse A^{-1}. Prove that $\left(A^{-1}\right)^* A^* = A^*\left(A^{-1}\right)^* = \mathrm{id}$. Deduce that A^* is invertible and that $(A^*)^{-1} = \left(A^{-1}\right)^*$.

6.3 Exercise 3 - Creation, annihilation and number

Consider the Hilbert space $\mathcal{H} := \ell^2(\mathbb{N})$.

a Define the operator A as

$$(A\alpha)_n = \alpha_{n+1} \qquad \forall n \in \mathbb{N}, \tag{43}$$

for any $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \in \mathcal{H}$.

Prove that A is a well defined linear bounded operator, find its norm and its spectrum.

b Consider A^* the adjoint of A. Show its explicit action and find its norm and its spectrum.

c Define $B := A^*A$. Prove that B is a self-adjoint operator, show its explicit action and find its norm and its spectrum.

Hint: Recall that if T is a linear bounded operator, the spectrum $\sigma(T)$ is a closed set, $\rho(T) \equiv \mathbb{C}\setminus\sigma(T)$ the resolvent of T is defined as

$$\rho(T) := \left\{\lambda \in \mathbb{C} \mid\ (T-\lambda\,\mathrm{id})^{-1} \text{ is a well-defined, linear, bounded operator}\right\}, \tag{44}$$

and that $\sigma(T) \subseteq \overline{B_{\|T\|}(0)}$, where $B_R(0) := \{\alpha \in \mathcal{H} \mid\ \|\alpha\|_2 < R\}$.

6.4 Exercise 4 - Operator norm of multiplication for a function

Consider the interval $I = (a,b) \subseteq \mathbb{R}$ and the Hilbert space $\mathcal{H} := L^2(I)$. Consider $\varphi \in C(I)$ a real valued continuous function with $\|\varphi\|_\infty < +\infty$. Consider the operator T_φ defined for any $\psi \in \mathcal{H}$ as

$$T_\varphi\psi(x) := \varphi(x)\,\psi(x)\,. \tag{45}$$

Prove that T_φ is a well defined linear bounded operator and prove that $\sigma(T_\varphi) = \overline{\varphi(I)}$.

Hint: Show first that $\varphi(I) \subseteq \sigma(T_\varphi)$ *and use the fact that the spectrum is closed to show that the same is true for the closures. Next, show that* $\left(\overline{\sigma(T_\varphi)}\right)^{\mathrm{c}} \subseteq \rho(T_\varphi)$ *to conclude.*

7 Exercise Sheet 7

7.1 Exercise 1 - Application of the UBP to the dual space

Let V be a Banach space and E a nonempty subset of V such that for any $\xi \in V^*$ there exists a finite constant C_ξ such that

$$\sup_{x \in E} |\xi(x)| \leqslant C_\xi. \tag{46}$$

Prove that E must be bounded.

Hint: Consider the map $J : V \to V^{**}$ defined as

$$[J(x)](\xi) := \xi(x) \qquad \forall x \in V,\ \xi \in V^*. \tag{47}$$

Prove that $\|J(x)\|_{V^{**}} = \|x\|$ for any $x \in V$. Use the Uniform Boundedness Principle to show that $J(E)$ is bounded and conclude.

7.2 Exercise 2 - Projection valued measures

Consider (X, Ω) a measurable space (i.e., a set X with a σ-algebra Ω in it), and consider a projection-valued measure with values in $\mathcal{H}$ an Hilbert space. Let $E,\ F \in \Omega$.

a Prove that if $E \cap F = \varnothing$ then $\operatorname{Ran} \mu(E) \perp \operatorname{Ran} \mu(F)$.

b Prove that $\mu(E)\mu(F)$ is an orthogonal projector and that

$$\operatorname{Ran}(\mu(E)\mu(F)) = \operatorname{Ran}\mu(E) \cap \operatorname{Ran}\mu(F). \tag{48}$$

7.3 Exercise 3 - $[A, B] = 0 \Rightarrow [f(A), B] = 0$

Let $\mathcal{H}$ be an Hilbert space. Let A be a self-adjoint bounded operator over $\mathcal{H}$. Let B a bounded operator over $\mathcal{H}$ such that $[A, B] = 0$. Consider a bounded complex-valued measurable function f. Prove that $[f(A), B] = 0$.

7.4 Exercise 4 - Norm and spectral radius

Let $\mathcal{H}$ be an Hilbert space. Let T be a bounded operator over $\mathcal{H}$. We proved in class that in general $R(T) \leqslant \|T\|$, where

$$R(T) := \sup_{\lambda \in \sigma(T)} |\lambda|. \tag{49}$$

Exhibit an explicit operator such that $R(T) < \|T\|$.

8 Exercise Sheet 8

8.1 Exercise 1 - Commuting operators and invertibility

a Let $\mathcal{H}$ be an Hilbert space. Suppose $A, B \in \mathcal{B}(\mathcal{H})$ with $[A, B] = 0$ and A not invertible. Prove that AB is not invertible.

Hint: Prove first that if AB were invertible then A would have both a left and a right inverse. Then prove that those would need to be equal and conclude.

b Prove that if we do not assume A and B to commute, the result in **a** is false.

8.2 Exercise 2 - An operator with a closed extension is closable

Let $\mathcal{H}$ be an Hilbert space. Let A be an unbounded linear operator on $\mathcal{H}$. Suppose there exists a closed operator C that extends the operator A. Prove that A is closable.

8.3 Exercise 3 - Explicit norm of resolvent operator

Let $\mathcal{H}$ be an Hilbert space. Let A be self-adjoint.

a Suppose $\lambda_0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Prove that

$$\left\| (A - \lambda_0 \,\mathrm{id})^{-1} \right\| = \frac{1}{d(\lambda_0, \sigma(A))}, \tag{50}$$

where $d(x, Y) := \inf_{y \in Y} |x - y|$, with $x \in \mathbb{C}$, $Y \subseteq C$.

Hint: Think of $(A - \lambda_0 \,\mathrm{id})^{-1}$ as a function of A in the sense of the functional calculus of A.

b Let $\lambda_0 \in \mathbb{C}$ and suppose that there exists $\varepsilon > 0$ and some nonzero $\psi \in \mathcal{H}$ such that

$$\| A\psi - \lambda_0 \psi \| < \varepsilon \|\psi\|. \tag{51}$$

Prove that there exists $\lambda \in \sigma(A)$ such that $|\lambda - \lambda_0| < \varepsilon$.

8.4 Exercise 4 - The delta is not a closable operator

Let $\mathcal{H} = L^2(I)$, with $I = [0, 1]$. Consider the operator A with domain $D(A) = C(I)$ and with action

$$A\psi(x) = \psi(0), \qquad \forall \psi \in D(A). \tag{52}$$

Prove that A is not closable.

9 Exercise Sheet 9

9.1 Exercise 1 - Hardy inequality

Let $k \in \mathbb{Z}$, $d \in \mathbb{N}$, $k + d \neq 0$. Let D be defined as

$$D := \begin{cases} C_c^\infty\left(\mathbb{R}^d\right) & \text{if } k \geqslant 0, \\ C_c^\infty\left(\mathbb{R}^d \setminus \{\mathbf{0}\}\right) & \text{if } k \leqslant -1,\ k+d \neq 0. \end{cases} \tag{53}$$

Prove that for any $\psi \in D$

$$\int_{\mathbb{R}^d} |\mathbf{x}|^k |\psi(\mathbf{x})|^2 \, d\mathbf{x} \leqslant \frac{4}{|k+d|^2} \int_{\mathbb{R}^d} |\mathbf{x}|^{k+2} |\nabla\psi(\mathbf{x})|^2 \, d\mathbf{x}. \tag{54}$$

Hint: Use the fact that

$$|\mathbf{x}|^k = \frac{1}{k+d} \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(|\mathbf{x}|^k x_j\right) \tag{55}$$

to integrate by part on the left hand side of (54) *and then use the Cauchy-Schwartz inequality.*

Remark: Notice that in particular if $k = -2$ (and $d \neq 2$) this implies that as operators

$$\frac{1}{|\mathbf{x}|^2} \leqslant -\frac{4}{|d-2|}\Delta. \tag{56}$$

A generalisation of this formula is called in the literature the **Hardy inequality**.

9.2 Exercise 2 - The Coulomb hamiltonian is self-adjoint

a Let $\mathcal{H} := L^2\left(\mathbb{R}^3\right)$. Define (as in class) the operator H_0 with[3]

$$\mathcal{D}(H_0) := H^2\left(\mathbb{R}^3\right) \equiv \left\{\psi \in \mathcal{H} \middle| \ |\mathbf{k}|^2 \widehat{\psi}(\mathbf{k}) \in L^2\left(\mathbb{R}^3\right)\right\}, \tag{57}$$

$$H_0\psi = -\Delta\psi = \left(|\mathbf{k}|^2 \widehat{\psi}(\mathbf{k})\right)^{\vee}, \qquad \forall \psi \in \mathcal{D}(H_0). \tag{58}$$

Prove that H_0 is closed.

b Let $\mathcal{D}(H) := \mathcal{D}(H_0)$. Define $H := H_0 + \frac{1}{|\mathbf{x}|}$. Prove that H is well-defined and closed. (Assume, if necessary, to know that there exists a positive constant C such that for any $\psi \in H^2\left(\mathbb{R}^3\right)$ it holds $\|\psi\|_{L^\infty} \leqslant C \|\psi\|_{H^2}$).

Hint: Use the fact that $H^2\left(\mathbb{R}^3\right) \subseteq L^\infty\left(\mathbb{R}^3\right)$ *to prove that is well-defined. To prove the closure, use* (54) *from Exercise 1 to show and subsequently use that* $\forall \varepsilon > 0$, $\forall \psi \in \mathcal{D}(H)$

$$\left\|\frac{1}{|\mathbf{x}|}\psi\right\|_{L^2} \leqslant \frac{2}{\varepsilon}\|\psi\|_{L^2} + \varepsilon\|H_0\psi\|_{L^2} \tag{59}$$

[3] Recall that we proved in the exercise session that if $\|\psi\|_{H^2} := \left\|\left(1+|\mathbf{k}|^2\right)\widehat{\psi}\right\|_{L^2}$, then $H^2\left(\mathbb{R}^3\right)$ is closed with respect to $\|\cdot\|_{H^2}$.

to get that

$$\|H_0\psi\|_{L^2} \leqslant \frac{2}{\varepsilon(1-\varepsilon)}\|\psi\|_{L^2} + \frac{1}{1-\varepsilon}\|H\psi\|_{L^2}. \tag{60}$$

c Prove that H is symmetric.

d Prove that H is self-adjoint.

Hint: Use the fact that $\frac{1}{|x|}$ *is a self-adjoint operator and apply the Kato-Rellich theorem.*

9.3 Exercise 3 - The square root is monotonous

Let $\mathcal{H}$ an Hilbert space and let $A,\ B \in \mathcal{B}(\mathcal{H})$, $A^* = A$, $B^* = B$

a Suppose[4] $A \geqslant \mathrm{id}$; prove that A is invertible with $A^{-1} \in \mathcal{B}(\mathcal{H})$ and that $0 \leqslant A^{-1} \leqslant \mathrm{id}$.

b Suppose $0 \leqslant A \leqslant B$; prove that for any $\lambda > 0$, $A+\lambda\,\mathrm{id}$ and $B+\lambda\,\mathrm{id}$ are invertible with $(A+\lambda\,\mathrm{id})^{-1},\ (B+\lambda\,\mathrm{id})^{-1} \in \mathcal{B}(\mathcal{H})$ and that we have $(B+\lambda\,\mathrm{id})^{-1} \leqslant (A+\lambda\,\mathrm{id})^{-1}$.

c Suppose $0 \leqslant A \leqslant B$; prove that $\sqrt{A} \leqslant \sqrt{B}$.

Hint: Prove and use the fact that

$$\sqrt{x} = \frac{1}{\pi}\int_0^{+\infty} \frac{1}{\sqrt{\lambda}}\left(1 - \frac{\lambda}{x+\lambda}\right) d\lambda, \qquad \forall x \geqslant 0. \tag{61}$$

9.4 Exercise 4 - Exercise on norm of the resolvent

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear self-adjoint operator on $\mathcal{H}$ with $A \geqslant 0$ and $\lambda > 0$. Denote with $\|\cdot\|$ the operator norm and with $\|\cdot\|_{\mathcal{H}}$ the norm induced by the inner product in the Hilbert space $\mathcal{H}$.

a Prove that $\left\|(A+\lambda\,\mathrm{id})^{-1}\right\| \leqslant 1/\lambda$.

b Prove that for all $\psi \in \mathcal{H}$,

$$\|\psi\|^2_{\mathcal{H}} \geqslant \left\|A(A+\lambda\,\mathrm{id})^{-1}\psi\right\|^2_{\mathcal{H}} + \lambda^2\left\|(A+\lambda\,\mathrm{id})^{-1}\psi\right\|^2_{\mathcal{H}}. \tag{62}$$

Conclude that $\left\|A(A+\lambda\,\mathrm{id})^{-1}\right\| \leqslant 1$.

[4] Recall that $A \geqslant 0$ if for any $\psi \in \mathcal{D}(A)$, $\langle\psi, A\psi\rangle \geqslant 0$ and that $A \geqslant B$ if $A - B \geqslant 0$.

10 Exercise Sheet 10

10.1 Exercise 1 - The generator of the translation is the momentum

Let $\mathcal{H} := L^2(\mathbb{R})$ and $P := -i\partial_x$ the momentum operator defined on the domain $\mathcal{D}(P) := H^1(\mathbb{R})$ as $P\psi(x) = -i\frac{\partial\psi}{\partial x}(x)$. Consider for any $\lambda \in \mathbb{R}$ the bounded operator T_λ defined for any $\psi \in \mathcal{H}$ as $T_\lambda\psi(x) = \psi(x-\lambda)$.

Prove that $\{T_\lambda\}_{\lambda\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group and that

$$T_\lambda = e^{i\lambda P} = e^{\lambda\partial_x}. \tag{63}$$

10.2 Exercise 2 - Condition for self-adjointness (complement to the class)

Let $\mathcal{H}$ be an Hilbert space, A a symmetric operator and $\mu > 0$ a positive real number. Prove that the following are equivalent.

a A is self-adjoint.

b $\operatorname{Ran}(A + i\mu\,\mathrm{id}) = \operatorname{Ran}(A - i\mu\,\mathrm{id}) = \mathcal{H}$.

10.3 Exercise 3 - Unitary operators as exponentials

Let $\mathcal{H}$ be an Hilbert space. Let $U \in \mathcal{B}(\mathcal{H})$. Prove that U is unitary if and only if there exist a self-adjoint operator A on $\mathcal{H}$ such that $U = e^{iA}$.

10.4 Exercise 4 - Bogoliubov diagonalization - part I

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \mathrm{id}, \tag{64}$$

$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{65}$$

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^*A_+ + A_-^*A_-\right) + \zeta\left(A_+^*A_-^* + A_+A_-\right). \tag{66}$$

a Prove that H is self-adjoint.

b Prove that there exist operators $C_\pm$ and numbers $\alpha,\ \beta \in \mathbb{R}$ such that

$$\left[C_\pm, C_\pm^*\right] = \mathrm{id}, \tag{67}$$

$$\left[C_+, C_-\right] = \left[C_+, C_-^*\right] = 0, \tag{68}$$

$$H = \alpha\left(C_+^*C_+ + C_-^*C_-\right) + \beta. \tag{69}$$

Hint: Define

$$C_\pm := \gamma_\pm A_\pm + \xi_\pm A^*_\mp \tag{70}$$

for some $\gamma_\pm,\ \xi_\pm \in \mathbb{R}$*. Use* (67) *and* (68) *to deduce that* $\gamma_+ = \gamma_-$, $\xi_+ = \xi_-$ *and that* $\gamma_\pm^2 - \xi_\pm^2 = 1$*. Calculate* $C^*_\pm C_\pm$ *and deduce* (69)*.*

11 Exercise Sheet 11

11.1 Exercise 1 - Double Harmonic oscillator

Let $\mathcal{H} = L^2\left(\mathbb{R}^2\right)$. Let $\widetilde{H}$ be defined as

$$\widetilde{H} := -\frac{1}{2}\left(\Delta_x + \Delta_y\right) + \frac{1}{2}\left(x^2 + y^2\right) - \lambda xy \tag{71}$$

with $D\left(\widetilde{H}\right) = C_c^\infty\left(\mathbb{R}^2\right)$.

Prove that if $\lambda \in (-1, 1)$ then $\widetilde{H}$ is essentially self adjoint and study the spectrum of the closure of $\widetilde{H}$.

Hint: Prove that, with the right change of variables $(x, y) \to (w, z)$, $\widetilde{H} = H_w + H_z$ with H_w only depending on w and H_z only depending on z.

11.2 Exercise 2 - Normal matrices polynomials

Let A be a normal matrix (meaning that $AA^* = A^*A$) and p a polynomial in two variables. Show by example that an eigenvector for $p(A, A^*)$ is not necessarily an eigenvector for A.

Remark: Even if eigenvectors of $p(A, A^*)$ do not correspond to eigenvectors of A, the spectrum does, in the sense that

$$\sigma\left(p\left(A, A^*\right)\right) = \left\{p\left(\lambda, \lambda^*\right) \mid \lambda \in \sigma(A)\right\}. \tag{72}$$

11.3 Exercise 3 - Spectral measure of the laplacian

Let $I := [0, 1]$ and consider $\mathcal{H} = L^2(I)$. Define the operator $H := -\Delta$ with domain[5] $D(H) := H^2(I) \cap C^1_{\text{per}}(I)$. Prove that H is self-adjoint and exhibit its spectral measure explicitly.

11.4 Exercise 4 - Bogoliubov diagonalization - part II

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \text{id}, \tag{73}$$

$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{74}$$

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^*A_+ + A_-^*A_-\right) + \zeta\left(A_+^*A_-^* + A_+A_-\right). \tag{75}$$

[5] This definition makes sense, because we know that for any function $\psi \in H^2(I)$ we have that there is a function $\widetilde{\psi} \in C^1(I)$ that coincides almost everywhere with ψ. The definition of the domain is then the set of functions $\psi \in H^2(I)$ such that the function $\widetilde{\psi}$ is periodic with derivative which is periodic.

Recall that if $\theta = \frac{1}{2}\operatorname{arctanh}\left(\frac{\zeta}{\eta}\right)$, $\alpha = \sqrt{\eta^2 - \zeta^2}$, $\beta = \sqrt{\eta^2 - \zeta^2} - \eta$ and C_+ and C_- are defined as

$$C_\pm := \cosh(\theta) A_\pm + \sinh(\theta) A_\mp^* \tag{76}$$

we get

$$\left[C_\pm, C_\pm^*\right] = \mathrm{id}, \tag{77}$$

$$\left[C_+, C_-\right] = \left[C_+, C_-^*\right] = 0, \tag{78}$$

$$H = \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta. \tag{79}$$

a Consider $X := A_+^* A_-^* - A_+ A_-$. Prove that X is skew-adjoint, meaning that $X^* = -X$.

b For any $t \in \mathbb{R}$ consider $U(t) := e^{-tX}$. Prove that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group such that

$$U(t) A_\pm U(-t) = \cosh(t) A_\pm + \sinh(t) A_\mp^*. \tag{80}$$

Hint: Consider for any $\psi, \varphi \in \mathcal{H}$ the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f_\pm(t) := \langle \psi, U(t) A_\pm U(-t) \varphi \rangle. \tag{81}$$

Prove that f satisfies a closed second order differential equation and deduce (80).

c Suppose that there is a complete orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that $A_\pm^* A_\pm \varphi_n = \epsilon_n^\pm \varphi_n$, with $\epsilon_n^\pm \in \mathbb{R}$. Prove that there exist a complete orthonormal system $\{\psi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that

$$H\psi_n = \left[\alpha\left(\epsilon_n^+ + \epsilon_n^-\right) + \beta\right]\psi_n. \tag{82}$$